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Stiffness Analysis Using Multi global Axes System

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Introduction

THE analysis of a structural system by the stiffness method can be achieved from the set of dependent equations

$$[S_c][\Delta_c] = [JL_c] + [R_c] \tag{1}$$

which describe the static equilibrium of the numerous joints within the structure: $[S_c]$ is the complete structure stiffness matrix, $[\Delta_c]$ is the complete joint displacement matrix, $[JL_c]$ is the complete joint load matrix and $[R_c]$ is the complete support reaction matrix. This matrix equation [Eq. (1)] yields the set of independent equations

$$[S_{uu}][\Delta_u] = [JL_u] \tag{2}$$

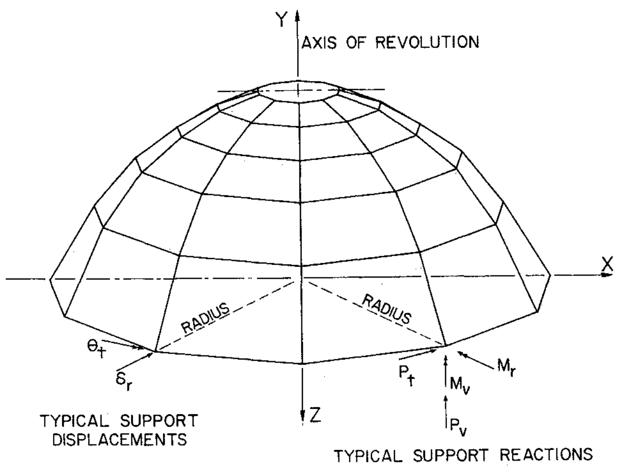

Fig. 1 Rigid framed dome.

which can be solved to evaluate the unrestrained components of joint displacement of the structure for specified loading conditions, and

$$[R_r] = [S_{ru}][S_{uu}]^{-1}[JL_u] - [JL_r] \tag{3}$$

which can be used to evaluate the support reactions of the structure. In Eqs. (2) and (3), $[S_{uu}]$ is the matrix of structure stiffness coefficients corresponding to the unrestrained components of joint displacement of the actual structure resulting from unit values of each of these components of joint displacements applied to the restrained structure; $[S_{ru}]$ is the matrix of structure stiffness coefficients corresponding to restrained components of joint displacement of the actual structure resulting from unit values of unrestrained components of joint displacements of the actual structure applied to the restrained structure; $[\Delta_u]$ is the matrix of unrestrained independent components of joint displacement; $[JL_u]$ and $[JL_r]$ are the matrices of joint loads corresponding to the unrestrained and restrained components of joint displacement, respectively; and $[R_r]$ is the matrix of the components of support reactions. The derivation and application of Eqs. (1–3) for the analysis of a structural system can be found in Refs. 1–3 and will not be discussed herein.

It has been general practice when developing the various matrices of Eq. (1) to describe all of the components of joint displacement, joint load and support reaction with respect to a single set of coordinate axes, i.e., the global or reference axes. Consequently, in order to maintain the continuity of Eq. (1) the structure stiffness coefficients which make up the matrix $[S_c]$ must be evaluated with respect to this set of reference axes; this means that the element stiffness coefficients which relate the end actions and the end displacements of an individual element, must be defined with respect to this single set of reference axes. Unfortunately, there are instances where it is either impossible or impractical to develop the set of independent joint equilibrium equations of Eq. (2) with respect to a single set of reference axes. Consider, for example, the rigid framed dome structure of Fig. 1. If it were supported at each joint of the lower ring such that the joints were free to translate radially in the plane of the ring (δ_r) and rotate about an axis in the plane of the ring perpendicular to the radius of the ring at the joint (θ_t), it would be most difficult to describe the components of displacement of all of the boundary joints with respect to a single set of rectangular Cartesian coordinate axes

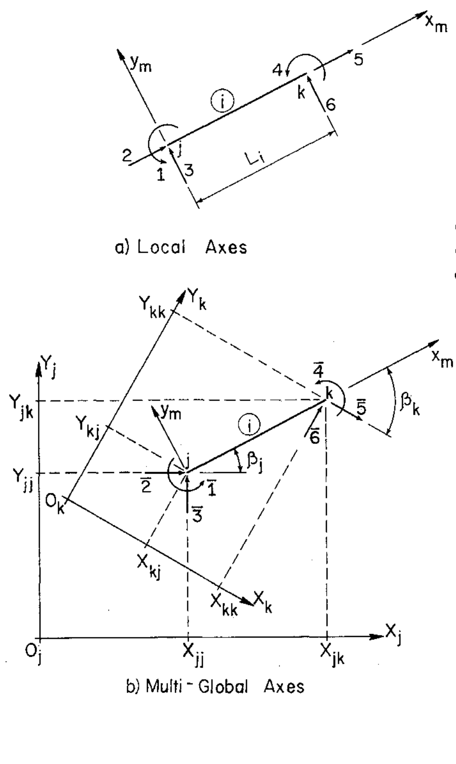

Fig. 2 Identification of axes and end actions or displacements for beam element.

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and obtain the set of independent equations defined by the matrix equation of Eq. (2) directly. For this particular structure a minimum of three different rectangular Cartesian coordinate global axes would be required in order to establish the independent components of unrestrained displacement of all of the joints of the system. Since the joints around the other rings of the dome are free of any external constraints their displacements can be described with respect to a single set of global axes or with respect to the same set of global axes that is required to identify the independent components of displacement of the support joint of the respective rib.

If, when developing the joint equilibrium equations with respect to a single set of rectangular coordinate axes, the components of displacement of one or more boundary joints are dependent upon one another, the set of equations represented by Eq. (2) would contain dependent equations. Hence the matrix $[S_{uu}]$ would be singular, i.e., $|S_{uu}| = 0$, and a solution could not be obtained. If the various dependent equations could be identified, the set of equations could be modified to yield a set of independent equations that would permit a solution. Unfortunately, this procedure is cumbersome when trying to develop a general computer program employing the stiffness method to analyze framed, mass or surface structures. The development of unwanted dependent equations can be avoided by using multi-global axes for the development of the joint equilibrium equations [Eq. (1)].

Analysis of Structural Elements

The primary task in developing the joint equilibrium equations [Eq. (1)] for the analysis of a structural system by the stiffness method is the evaluation of the structure stiffness coefficients which are the elements of the complete structure stiffness matrix $[S_c]$ of Eq. (1). These coefficients are best determined by a process of summation of the element stiffness coefficients of the various elements of the structure.

For the purpose of discussion consider the analysis of a typical beam element of a planar framed structure. Now, the possible independent components of end action or end displacement can be defined with respect to a set of rectangular coordinate axes peculiar to the individual element, i.e., the local axes, such that the x_m axis defines the longitudinal axis of the beam and the y_m axis rotated counterclockwise from the x_m axis defines a principal axis of the beam cross section; the origin of the local axes is taken at the j end of the beam; the possible components of end action and end displacement in the frame of the local axes are identified and ordered as shown in Fig. 2a. The element stiffness matrix for the beam element is defined as $[K]_i$; this matrix defines the relationship between the components of end actions and of end displacement with respect to the local axes of the element, i.e.,

$$[M]_i = [K]_i[\delta]_i \tag{4}$$

where $[M]_i$ is the matrix of the components of end actions with respect to the local axes and $[\delta]_i$ is the matrix of the components of end displacement with respect to the local axes (in the order of their labels).

The displacements of the ends of element i are controlled by the displacements of the joints that the element frames into; and the displacements of the joints are defined with respect to the system of global axes. Thus, the displacement of each end of element i needs to be described in terms of components in the frame of the global axes used to define the corresponding joint displacement; the end actions at each end of the element can be described in terms of components in the frame of the same global axes. Now, the components of end displacement or end action which have been defined with respect to the local axes at each end of the element i can be resolved into components in the frame of a particular set of global axes by means of a rotation matrix. Assuming that the end actions and displacements at the j-end of element i are to be defined with respect to the set of global axes $X_j - Y_j$ and that the end actions and displacements at the k-end of the element are to be defined with respect to the set of global axes $X_k - Y_k$ as indicated in Fig. 2b, the components of end action or displacement in the frame of the local axes of element i and in the frame of the related global axes are related by the transformation matrix

$$[T]_i = \left[\begin{array}{c|c} [C_j]_i & [0] \\ \hline [0] & [C_k]_i \end{array}\right]_i \tag{5}$$

where $[C_j]_i$ and $[C_k]_i$ are the rotation matrix for the j and k ends of element i, respectively. Defining the rotation matrices in terms of direction cosines, Eq. (5) can be written as

$$[T]_i = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{jx} & C_{jy} & 0 & 0 & 0 \\ 0 & -C_{jy} & C_{jx} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{kx} & C_{ky} \\ 0 & 0 & 0 & 0 & -C_{ky} & C_{kx} \end{array}\right]_i \tag{6}$$

where

$$C_{jx} = \cos\beta_j = X_{jk} - X_{jj}/L_i \tag{7}$$

$$C_{jy} = \cos(90 - \beta_j) = \sin\beta_j = \frac{Y_{jk} - Y_{jj}}{L_i} \tag{8}$$

$$C_{kx} = \cos\beta_k = \frac{X_{kk} - X_{kj}}{L_i} \tag{9}$$

and

$$C_{ky} = \cos(90 - \beta_k) = \sin\beta_k = \frac{Y_{kk} - Y_{kj}}{L_i} \tag{10}$$

(X_{jj},Y_{jj}), (X_{jk},Y_{jk}), (X_{kj},Y_{kj}) and (X_{kk},Y_{kk}) are the coordinates of the j and k ends of the element i with respect to the two global axes systems (the first subscript identifies the global axes system as either the $X_j - Y_j$ or $X_k - Y_k$ system and the second subscript identifies the end of the member as either the j or k end) and L_i is the length of the element. The angle β is defined as the angle between X axis of a global system of axes and the x_m axis of the local system of axes for the element, measuring the angle counterclockwise from the X axis of the global system.

Now the components of end actions $[M]_i$ and end displacements $[\delta]_i$ in the frame of the local axes can be resolved into their respective components in the frame of the related multi-global axes by means of the matrix equations

$$[\bar{M}]_i = [T]_i^T[M]_i \tag{11}$$

and

$$[\bar{\delta}]_i = [T]_i^T[\delta]_i \tag{12}$$

where $[T]_i^T$ is the transpose of the transformation matrix of Eq. (5). Combining Eqs. (4), (11) and (12) the element stiffness matrix $[K]_i$ for a beam element i written with respect to the local axes can be transformed into the frame of the related multi-global axes by the matrix equation

$$[\bar{K}]_i = [T]_i^T[K]_i[T]_i \tag{13}$$

where $[\bar{K}]_i$ is defined as the transformed element stiffness matrix for element i. Once the transformed element stiffness matrix has been derived for each of the elements of the structure, the complete structure stiffness matrix can be developed. The transformation matrix $[T]_i$ can be developed for any structural element i to relate the local axes and a system of multi-global axes.

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Pressure Waves for Flow Induced Acoustic Resonance in Cavities

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Introduction

FOR several years, considerable interest has been directed toward understanding separated flow, primarily because of its unavoidable occurrence in many aerodynamic configurations.[1] A particular type of separated flow is herein considered, namely, that caused by cavities in aerodynamic surfaces. As pointed out in a review article,[1] the concern of previous investigators into cavity flow has mainly centered around the effects of such flow on drag and heat transfer during steady flow conditions. The situation related to steady cavity flow is reasonably well understood, subject to certain geometrical constraints.

More recently, it has been observed by several investigators[2–5] that cavities will emit acoustic radiation of a sharply defined frequency under proper conditions of external flow and cavity geometry. Relatedly, resonance is noted to significantly alter the drag[6] and heat transfer[4] characteristics of cavities in comparison to their nonresonating performance. To date, very little experimental data is available which would enable one to clearly define the nature of and the requisite conditions for flow-induced acoustic resonance; nor is there any general theoretical agreement concerning these matters.

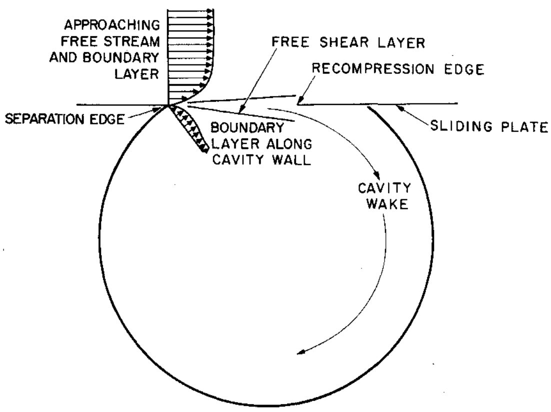

Fig. 1 Cavity flow model.

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This present Note describes an experimental investigation into the pressure fields existing in cavities during situations of flow-induced resonance, and it represents a portion of a larger study into cavity resonance carried out by coauthor G. H. Watson during his doctoral dissertation.[7] The reported pressure measurements were taken in a near-circular cylinder of 12 in. diam, while the external flow velocity ranged from about 30 to 200 fps. Interpretation of the data allows for classifying each instance of resonance into one of three types. Additionally, all the measured resonant frequencies are correlated by a Strouhal number based on cavity diameter.

Experimental Investigation

The cavity model employed was a circular cylinder of 12 in. diam 12 in. length (normal to the flow), as depicted in Fig. 1. The portion of the cavity adjacent to the freestream was sliced off, resulting in a 9 in. depth and a maximum cavity opening (mixing length) of 8.75 in. The mixing length could be adjusted to any value less than its maximum value by means of the indicated sliding plate, which was constructed from $\frac{1}{16}$-in. steel plate and had a sharpened upstream edge. One end of the cavity model, constructed of $\frac{1}{4}$-in. aluminum plate, had 23 threaded holes located as shown to scale by the crosses in Figs. 2–4. These holes accepted either flush mounted pressure transducers or blank plugs.

The primary flow was through a 1 × 1 ft nozzle exhausting to the atmosphere following an 8:1 contraction. The plate just upstream of the cavity was actually one wall of the nozzle, and it was equipped with a perforated surface which allowed for boundary-layer removal or augmentation. The influence of the condition of the boundary layer at separation will not be herein considered, although Watson[7] did consider it in detail; suffice it to say that thin boundary layers at separation promote both the occurrence and the intensity of resonance.

The existence of flow-induced resonance in the cavity could be ascertained simply by noting the sound generated. More quantitatively, an oscilloscope tracing from one of the aforementioned pressure transducers during a quiescent condition revealed a random pressure fluctuation within the cavity of a relatively small amplitude; conversely, when the cavity was in a resonant condition the pressure variation was distinctly sinusoidal and of a much larger value. The controllable experimental parameters which were found to influence the occurrence of resonance are free-stream velocity, mixing length, and boundary-layer thickness at separation.

The testing procedure developed for the present purposes consisted of setting the boundary-layer control and the mixing length to particular values and then gradually increasing the freestream velocity until a sharply tuned condition of resonance was obtained, as indicated by a pressure transducer output. Then a mapping of the resulting pressure field was obtained by simultaneously monitoring the output from two

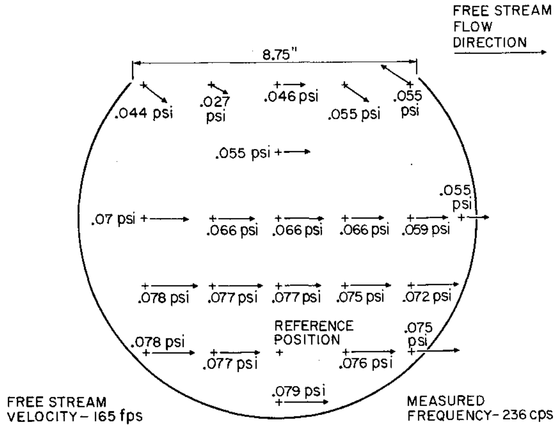

Fig. 2 Polar plot of acoustic pressure within cavity (Helmholtz mode).